LECTURE NOTES: FUNDAMENTAL ALGEBRAIC GEOMETRY, SUMMER SCHOOL, ADDIS ABABA

0.1. **Proem.** These are lecture notes for a summer school in algebraic geometry, held at the University of Addis Ababa in August 2026. They are not finished - towards the end they become somewhat sketchy. I aim to continue updating the notes throughout the week. In addition, there is almost certainly more material in here than we will have time to discuss.

1. Lecture 1: Affine varieties

We start out by constructing the basic objects of interest in algebraic geometry.[1] First, we need a space to have our geometric objects in:

1.1. Affine space.

Definition 1.1. Let $\mathbb{A}^n_k$ be the n-dimensional vector space over a field k. We call $\mathbb{A}^n_k$ *n-dimensional affine space* over k.

If the base field k is clear from context, we will just write $\mathbb{A}^n$.

(We can have $n = 0$, in which case $\mathbb{A}^0$ is just a point. But there's not much interesting geometry to do with just a point.)

Often I'm going to use $k = \mathbb{C}$, because this will make the theory simpler. If you're familiar with *algebraically closed fields*, you can replace $\mathbb{C}$ by "an algebraically closed field" everywhere in these notes. [2] (Many algebraic geometers only work over $\mathbb{C}$ anyway, and if you like, every time I write "k algebraically closed", you can replace it with "$k = \mathbb{C}$".)

So that's a space in which to do geometry. If we choose a point $p \in \mathbb{A}^n$, then we can write $p = (a_1, \ldots, a_n)$ where each $a_i \in k$. We say that the a_i are the coordinates of p.

Consider now the ring of polynomials $k[x_0, \ldots, x_n]$. We will think of these polynomials as *functions* $\mathbb{A}^n \to \mathbb{A}^1 = k$: if $f(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]$ and $p = (a_1, \ldots, a_n)$, we simply set

$$f(p) = f(a_1, a_2, \ldots, a_n).$$

[1]This is largely taken from Chapter 1 of *Algebraic Geometry*, Robin Hartshorne, Springer 1977, and *Algebraic Geometry: An Introduction*, Daniel Perrin, Springer 1995

[2]A field k is algebraically closed if, for every $f(x) \in k[x]$, there is an $a \in k$ such that $f(a) = 0$. Exercise: show that if k is algebraically closed, every polynomial $f(x) \in k[x]$ can be written as a product of linear polynomials.

1.2. **Algebraic sets.** Let $S \subset k[x_1, \ldots, x_n]$ be some set of polynomials. The *algebraic set* $Z(S)$ determined by S is

Definition 1.2.

$$Z(S) = \{p \in \mathbb{A}^n \text{ such that } f(p) = 0 \text{ for every } f \in S\}.$$

Any set $V \subset \mathbb{A}^n$ for which there is some set $S \subset k[x_1, \ldots, x_n]$ such that $V = Z(S)$, we call an *algebraic set.*

Example 1.3. Any point $p \in \mathbb{A}^n$ is an algebraic set.

If we let $I(S) \subset k[x_1, \ldots, x_n]$ be the ideal generated by S, then we see (remember that any element of $I(S)$ is, by definition, a sum of terms each of which has some element of S as a factor) that

$$Z(S) = Z(I(S)).$$

So it's enough to work with ideals instead of with arbitrary sets of polynomials.

Example 1.4. Different ideals of $k[x_1, \ldots, x_n]$ can give the same algebraic set. For instance:

In $\mathbb{R}[x]$, let $I_1 = (x^2 + 1)$. Then $Z(I_1) = \emptyset = Z(1)$.

For another example, if $f \in k[x_1, \ldots, x_n]$, then $Z(f) = Z(f^n)$ for any positive n.

Let's look at how algebraic sets behave:

Proposition 1.5.
(1) *The union of two algebraic sets is an algebraic set.*
(2) *Any intersection of algebraic sets is an algebraic set.*
(3) *The empty set $\emptyset$ and affine space $\mathbb{A}^n$ are both algebraic sets.*

Proof. For the first: Let $V = Z(I), U = Z(J)$ be algebraic sets, then $V \cup U = Z(IJ)$.

For the second: Let $\{V_\alpha = Z(I_\alpha)\} : \alpha \in A$ be a collection of algebraic sets. Then $\bigcap V_\alpha = Z(\bigcup I_\alpha)$.

For the third: $\emptyset = V(1)$, $\mathbb{A}^n = V(0)$. $\square$

We see that this is precisely the set of axioms required for the closed sets of a topology!

Definition 1.6. The *Zariski* topology on $\mathbb{A}^n$ is the topology given by taking the algebraic sets to be the closed sets.

So now we can talk about closed sets and open sets (complements of closed sets).

We should note that the Zariski topology is in many ways a strange topology – it is not even Hausdorff in general!

Example 1.7. Let's look at the Zariski topology on $\mathbb{A}^1$:

First, we note that any polynomial $f(x) \in k[x]$ has a finite number of zeroes - therefore, any closed set except $\mathbb{A}^1$ itself is a finite set of points.

But it is also true that any singleton set $\{t\} \subset \mathbb{A}^1$ is closed, since it is the zero set of $x - t$.

It follows that the closed subsets of $\mathbb{A}^1$ are precisely the finite sets, and in this case, the Zariski topology equals the *finite complement topology.* (In particular, it's not Hausdorff if k is infinite.)

Example 1.8. Let $k = \mathbb{F}_{p^n}$ be a finite field. Then in the Zariski topology on $\mathbb{A}^n_k$, every set is both open and closed.

We can now make the following definition:

Definition 1.9. A nonempty subset Y of a topological space X is *irreducible* if there is no way of writing $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are proper subsets of Y, both closed in Y.

A subset that is not irreducible is called *reducible.*

(By convention, $\emptyset$ is reducible.)

Example 1.10. The field k is infinite if and only if $\mathbb{A}^1_k$ is irreducible.

The "cross" $Z(xy) \subset \mathbb{A}^2_k$ is reducible: it is the union of the two axes $Z(xy) = Z(x) \cup Z(y)$.

Any open subset of an irreducible set is irreducible.

We can now finally define what algebraic geometers are interested in:

Definition 1.11. An *affine variety* is an irreducible closed subset of $\mathbb{A}^n$.

An open subset of an affine variety is a *quasi-affine variety.*

Let's look some more on algebraic sets and ideals:

First, we need a way to go from an algebraic set to an ideal.

Definition 1.12. Let $Z \subset \mathbb{A}^n$, and let

$$I(Z) = \{f \in k[x_1, \ldots, x_n] \text{ such that } f(p) = 0 \text{ for every } p \in Z\}.$$

We call $I(Z)$ the *ideal of the set Z.*

Exercise 1.13. Show that $I(Z)$ indeed is an ideal of the ring $k[x_1, \ldots, x_n]$.

Then we have the following results:

Proposition 1.14.

If $I \subset J$ are ideals of $k[x_1, \ldots, x_n]$, then $Z(J) \subset Z(I)$.

If $Z_1 \subset Z_2$ are subsets of $\mathbb{A}^n$, then $I(Z_2) \subset I(Z_1)$.

Suppose $Z_1, Z_2 \subset \mathbb{A}^n$, then $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.

Proof. Exercise! $\square$

and

Proposition 1.15. *For any $Y \subset \mathbb{A}^n$, we have $Z(I(Y)) = \overline{Y}$, the* closure *of Y. (The closure of Y, written $\overline{Y}$, is defined as the intersection of all closed sets containing Y. Because any intersection of closed sets is closed, $\overline{Y}$ is closed.)*

Proof. In one direction, by definition $Z(I(Y))$ is closed, so we must have $\overline{Y} \subset Z(I(Y))$. In the other, let $p \in \overline{Y}$: then for every closed set W with $Y \in W$, we have $p \subset W$. Because W is closed, there is some ideal I_W such that $W = Z(I_W)$. Then, by proposition 1.14 (2), $I(W) \subset I(Y)$. But it's clear that $I_W \subset I(W)$, so by proposition 1.14 (1), $W = Z(I_W) \supset Z(I(Y))$.

But then $Z(I(Y))$ is a closed subset of every closed set containing Y, so we have $Z(I(Y)) \subset \overline{Y}$. $\square$

More difficult is the following:

Proposition 1.16. *Assume that $k = \mathbb{C}$. Then, if $I \subset k[x_1, \ldots, x_n]$ is an ideal, then $Z(I) = Z(\sqrt{I})$, where $\sqrt{I}$ is the* radical *of I*[3].

This follows immediately from Hilbert's *Nullstellensatz*, which I state below (but won't prove).

Theorem 1.17 (Hilbert's Nullstellensatz, Hilbert 1893)**.** *Let $k = \mathbb{C}$, let $I \subset A = k[x_1, \ldots, x_n]$, and let $f \in A$ such that $f(p) = 0$ for all $p \in Z(I)$. Then there is some integer $r \geq 1$ such that $f^r \in I$.*

Let's sum up:

To every subset $Y \subset \mathbb{A}^n$, we can associate an ideal $I(Y) \subset k[x_1, \ldots, x_n]$. In the other direction, to every ideal $I \subset k[x_1, \ldots, x_n]$, we can associate an algebraic set $Z(I) \subset \mathbb{A}^n$. If Y is an algebraic set, then $Z(I(Y)) = Y$. If $k = \mathbb{C}$ and $I = \sqrt{I}$, then $I(Z(I)) = I$.

1.3. **More about closed and open sets.** So now we know how to pass between ideals and algebraic sets. Can we characterise the ideals that correspond to affine varieties?

Corollary 1.18. *If a nonempty algebraic set $Y \subset \mathbb{A}^n$ is irreducible, its ideal $I(Y)$ is a prime ideal.*

Suppose that $k = \mathbb{C}$: Then, if $I \subset k[x_1, \ldots, x_n]$ is prime, $Z(I)$ is irreducible.

[3]The radical $\sqrt{I}$ of an ideal I is the set of all elements $f \in R$ such that for some $k > 0$, $f^k \in I$. The radical of an ideal is an ideal (exercise!)

Proof. Suppose that Y is irreducible, and let $f, g \in I(Y)$. Then $Y \subset Z(fg) = Z(f) \cup Z(g)$, so

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g)),$$

both of which are closed subsets of Y. But Y is irreducible, so we must have $Y = Y \cap Z(f)$ or $Y = Y \cap Z(g)$. Let's assume the first: in this case $Y \subset Z(f)$, so $f \in I(Y)$. In the other case $g \in I(Y)$. But then $I(Y)$ is prime.

Second, assume that $k = \mathbb{C}$: suppose $I = \mathfrak{p}$ is a prime ideal, and suppose $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. But then $Z(\mathfrak{p}) = Y_1$ or Y_2, so it's irreducible. □

Example 1.19. If $k = \mathbb{C}$, then $\mathbb{A}^n_k$ is irreducible, because $\mathbb{A}^n = Z((0))$, and the ideal (0) is prime.

Example 1.20. Let $k = \mathbb{C}$, and let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial of degree d. Then, because $k[x_1, \ldots, x_n]$ is a unique factorisation domain (UFD), the ideal (f) generated by f is prime, and $Z((f))$ is an affine variety.

If $n = 2$, we call $Z(f)$ a *curve*, if $n = 3$ a *surface*, for any bigger n a *hypersurface.* We say that d is the *degree* of the curve/surface/hypersurface $Z(f)$.

Example 1.21. Over $\mathbb{R}$, consider the polynomial $f(x, y) \in \mathbb{R}[x, y]$ given by $y^2 + x^2(x-1)^2$. Then f is irreducible (exercise!), so, since $\mathbb{R}[x, y]$ is a UFD, $(f) \subset \mathbb{R}[x, y]$ is a prime ideal. But $Z(f) = \{(0, 0), (1, 0)\}$ (exercise!), which is clearly reducible.

So it really is necessary to have $k = \mathbb{C}$ (or algebraically closed) in the results above.

Here's an important result that I won't prove:

Proposition 1.22. *Every algebraic set can be written uniquely as a finite union of affine varieties, none contained in each other.*

This shows that it really is the algebraic varieties that are fundamental.

1.4. **Dimension and the affine coordinate ring.** We'd like to be able to speak about the *dimension* of a variety. Intuitively, a point should have dimension 0, a curve dimension 1, and $\mathbb{A}^n$ dimension n.

In order to do that, we first need to introduce the following fact:

Lemma 1.23. *The Zariski topology on $\mathbb{A}^n$ is* noetherian*, which means that for every descending chain of closed sets*

$$Z_1 \supset Z_2 \supset \cdots$$

there is some r such that $Z_r = Z_{r+1} = Z_{r+2}$... We say that the chain terminates.

This is what we need to define the dimension.

Definition 1.24. Let X be an affine or quasi-affine variety.

The dimension of X, which we write $\dim X$, is the *supremum* of all integers n for which there is a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of distinct irreducible closed subsets of X.

Example 1.25. $\dim \mathbb{A}^1 = 1$, because the maximal chains of irreducible closed subsets of $\mathbb{A}^1$ are $\{p\} \subset \mathbb{A}^1$.

This dimension can of course be characterised algebraically as well.

To do that, first we need the following:

Definition 1.26. Let $X \subset \mathbb{A}^n$ be an affine algebraic set. Let $\mathcal{O}(X) = k[x_1, \ldots, x_n]/(I(X))$. We call $\mathcal{O}(X)$ the *affine coordinate ring* on X.

Definition 1.27. The *Krull dimension* (or just dimension) of a ring A is the supremum of the integers n such that there exists a chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n$$

of distinct prime ideals of A.

Lemma 1.28. *Let $X \subset A^n$ be an affine variety. Then* $\dim X = \dim \mathcal{O}(X)$.

Proof. The closed irreducible subsets of X correspond to prime ideals of $k[x_1, \ldots, x_n]$ containing $I(X)$. But these correspond 1-1 to prime ideals of $\mathcal{O}(X)$. So $\dim X$ is the length of the longest chain of prime ideals in $\mathcal{O}(X)$. But that is precisely the dimension. $\square$

Unfortunately, to use this properly, you need to do some fairly gnarly commutative algebra. So I'll just sum up the useful results:

Corollary 1.29.
(1) $\dim \mathbb{A}^n = n$.
(2) *If X is a quasi-affine variety,* $\dim \overline{X} = \dim X$.
(3) *If $X \subset \mathbb{A}^n$ is a variety and* $\dim X = n-1$, *then X is a hypersurface, i.e., $X = Z(f)$ for some irreducible $f \in k[x_1, \ldots, x_n]$.*
(4) *If $k = \mathbb{C}$, then for any irreducible $f \in k[x_1, \ldots, x_n]$ $Z(f) \subset \mathbb{A}^n$ is a hypersurface.*

2. Lecture 2: Projective varieties

Affine space is nice and easy to work with, but there are some problems with it. Consider the case of two distinct lines in $\mathbb{A}_k^2$: either they intersect in precisely one point, or they are parallel. Wouldn't it be nicer if it was impossible for two lines to be parallel, so that we could always say that two distinct lines intersect in a unique point? How might we do that?

The answer is to introduce projective space. We'll do it in two ways:

2.1. Projective space.

Consider the vector space k^{n+1}, and let $\mathbf{0}$ be the origin. Let's index the coordinates from 0 to n. We can define the following equivalence relation on $k^{n+1} \setminus \{\mathbf{0}\}$:

$$(a_0, a_1, \dots a_n) \sim (b_0, b_1, \dots, b_n)$$

if there exists some $\lambda \in k$, $\lambda \neq 0$, such that

$$(a_0, a_1, \dots a_n) = (\lambda b_0, \lambda b_1, \dots \lambda b_n).$$

Exercise 2.1. Use the axioms of equivalence relations to check that $\sim$ really *is* an equivalence relation.

Definition 2.2 (Projective space, definition 1)**.** We define $\mathbb{P}^n$, the *projective space of dimension n*, to be the set of equivalence classes of $k^{n+1} \setminus \mathbf{0}$ under the equivalence relation described above.

That's one way of thinking about it, but let's do it in a different way: Suppose that $p = (a_0, a_1, \dots a_n), q = (b_0, b_1, \dots, b_n)$ are two different points in k^{n+1}, and suppose that the line connecting p and q also passes through the origin $\mathbf{0}$: This means that $p \sim q$ under the equivalence relation above. On the other hand, suppose that the line connecting p and q does *not* pass through $\mathbf{0}$, then $p \not\sim q$.

Exercise 2.3. Prove the last claim.

This means that **every line in k^{n+1} passing through $\mathbf{0}$ corresponds to a unique $\sim$-equivalence class**.

This lets us say:

Definition 2.4 (Projective space, definition 2)**.** We define $\mathbb{P}^n$, the *projective space of dimension n*, to be the set of all lines in k^{n+1} passing through the origin.

Let's note that this makes it clearer why $\mathbb{P}^n$ should have dimension n: clearly k^{n+1} has dimension $n+1$, but when we contract every line through the origin, we subtract the dimension of the line - namely 1.

2.2. **Homogeneous coordinates.** The first definition allows us to describe points of $\mathbb{P}^n$ using coordinates: if $p = (a_0, a_1, \dots a_n)$ is a point in $k^{n+1} \setminus \{\mathbf{0}\}$, let us write its equivalence class as

$$\overline{p} = (a_0 : a_1 : \cdots : a_n).$$

By definition of the equivalence relation $\sim$, we can also write $\overline{p} = (\lambda a_0 : \lambda a_1 : \cdots : \lambda a_n)$ for any nonzero λ. We say that $(a_0 : a_1 : \cdots : a_n)$ are the *homogeneous coordinates* of $\overline{p}$.

Example 2.5. In $\mathbb{P}^2$, $(1 : 3 : 2) = (-2 : -6 : -4) = (50 : 150 : 100)$.

We can already describe some projective spaces:

Example 2.6. $\mathbb{P}^0$ is a point. Let's use our first definition of $\mathbb{P}^n$ to see this: any two points p, q in $k^1 \setminus \{\mathbf{0}\}$ are given by just $p = (a_0),\ q = (b_0)$, which means that $a_0 = \frac{a_0}{b_0} b_0$. So any two points are equivalent, and there's just one equivalence class.

We can also see this with our second definition: There's only one line in k^1 through the origin, so $\mathbb{P}^0$ is a point.

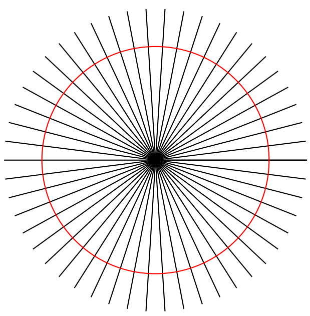
Let's now consider $\mathbb{P}^1$: this is more interesting. Let $p = (a_0 : a_1) \in \mathbb{P}^1$ be a point. There are now two cases: Either a_0 is nonzero or not. Suppose $a_0 \neq 0$: then there are no restrictions on a_1, and we can write

$$p = (a_0/a_0 : a_1/a_0) = (1 : a_1/a_0)$$

. If $a_0 = 0$, we have $p = (0 : a_1)$, so $a_1 \neq 0$. But then $p = (0 : 1)$.

So in the first case, we get an entire $k = \mathbb{A}^1$ of points, and there's just one point in the second case. So we can think of $\mathbb{P}^1$ as $\mathbb{A}^1$ with an additional point.[4]

Let's use our second definition, in the case $k = \mathbb{R}$:



The circle intersects each line twice, in opposite points. This means that $\mathbb{P}^1_{\mathbb{R}}$ is the space that we get from identifying opposite points on the circle. Convince yourself that this just gives you a circle again.

[4] If you're familiar with the Riemann sphere, this means that $\mathbb{P}^1_{\mathbb{C}}$ "is" the Riemann sphere. The idea of thinking of a projective line as a sphere unsettled me a bit when I was learning about projective space, but it turns out to be very useful.

How can we identify these two descriptions? Consider a line through the origin: then, unless the line is the y-axis, it has a *slope* $t \in \mathbb{R}$, and this slope uniquely determines the line. But then we can identify the set of lines with $\mathbb{R} \cup \{\textbf{a point}\}$, by sending any line to its slope and the y-axis to the other point.

2.3. **Homogeneous polynomials.** For affine space, we defined an algebraic set to be the set of points in $\mathbb{A}^n$ where some set of polynomials all vanish. How might we do something similar in $\mathbb{P}^n$? Suppose that we tried to define the zero set of the polynomial $f(x, y, z) = x - y^2 + z$ in $\mathbb{P}^2$: we find that $f(1, 1, 0) = 0$, and $f(2, 2, 0) = -2$. But since $(1 : 1 : 0) = (2 : 2 : 0)$ in $\mathbb{P}^2$, this means that we *can't* define $f((1 : 1 : 0))$.

The solution is to restrict ourselves to a smaller class of polynomials.

Definition 2.7. A polynomial $f \in k[x_0, x_1, \ldots, x_n]$ is *homogeneous* if all its terms have the same degree.

For example, $x - y,\ x^2 + y^2 + yz,\ x^3 - y(z - y)(z - x)$ are all homogeneous polynomials of $k[x, y, z]$.

Suppose now that $f \in k[x_0, x_1, \ldots, x_n]$ is a homogeneous polynomial of degree d. Let $a = (a_0, \ldots, a_n)$, choose some $\lambda \in k, \lambda \neq 0$, and set $b = (\lambda a_0, \ldots, \lambda a_n)$, which is to say that a, b represent the same point in $\mathbb{P}^n$. Then

$$f(b) = f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n) = \lambda^d f(a).$$

This means that *if $a \sim b$, for any homogeneous polynomial f, $f(a) = 0$ implies $f(b) = 0$.* So we can at least ask if a *homogeneous* polynomial vanishes at a point in $\mathbb{P}^n$. Therefore we can make the following definition:

Definition 2.8. Let $S \subset k[x_0, \ldots, x_n]$ be a set of homogeneous polynomials. The *algebraic set* $Z(S)$ is defined by

$$Z(S) = \{p \in \mathbb{P}^n | f(p) = 0 \text{ for all } f \in S\}.$$

Let us say that an ideal $I \subset k[x_0, \ldots, x_n]$ is *homogeneous* if it can be generated by homogeneous polynomials.

If $I \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal, we define

$$Z(I) = \{p \in \mathbb{P}^n | f(p) = 0 \text{ for all homogeneous } f \in I\}.$$

A subset $Y \subset \mathbb{P}^n$ is an algebraic set if it can be written as $Y = Z(S)$ for some set S of homogeneous polynomials.

Example 2.9. Let $f(x_0, x_1) = x_0 - 2x_1 \in k[x_0, x_1]$. Then f is a homogeneous polynomial of degree 1. We can then find $Z(f) \subset \mathbb{P}^1$, by solving $f(x_0, x_1) = 0$. If $p = (p_0 : p_1) \in \mathbb{Z}(f)$, it follows that $p_0 = 2p_1$. But this defines a point $(2 : 1) = (4 : 2) = (-100 : -50) \in \mathbb{P}^1$.

Example 2.10. Let $f(x_0, x_1) = x_0(x_0^2 - x_1^2) \in k[x_0, x_1]$. Then f is a homogeneous polynomial of degree 3. We find $Z(f) \subset \mathbb{P}^1$: A point $p = (p_0, p_1) \in Z(f)$ must then satisfy *either* $p_0 = 0$ or $p_0 - p_1 = 0$ or $p_0 + p_1 = 0$. But each of these linear factors determine a point in $\mathbb{P}^1$: the three points are $(0 : 1)$ and $(1 : 1)$ and $(1 : -1)$.

Example 2.11. Let $f(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2$, and $g(x_0, x_1, x_2) = x_0x_1x_2$. Then f is homogeneous of degree 2, g homogeneous of degree 3, both in $\mathbb{C}[x_0, x_1, x_2]$.

Let's try to find all the points in $X = Z(f, g) \subset \mathbb{P}^2_{\mathbb{C}}$. We see that if $p = (p_0, p_1, p_2) \in X$, we must have $p_0p_1p_2 = 0$, so at least one of p_0, p_1, p_2 is 0.

Let's assume that $p_0 = 0$: Then, since $f(p) = 0$, we must have $p_1^2 + p_2^2 = 0$. But this means that $p_2 = ip_1$ or $p_2 = -ip_1$. But this gives two points in $\mathbb{P}^2_{\mathbb{C}}$: $(0 : 1 : i)$ and $(0 : 1 : -i)$.

If we do the same computations for $p_1 = 0$ and $p_2 = 0$, we find four other points: $(1 : 0 : i), (1 : 0 : -i), (1 : i : 0), (1 : -i : 0)$. So in total, we find

$$X = \{(0 : 1 : i), (0 : 1 : -i), (1 : 0 : i), (1 : 0 : -i), (1 : i : 0), (1 : -i : 0)\}.$$

Lemma 2.12.
(1) *The union of two projective algebraic sets is a projective algebraic set.*
(2) *Any intersection of projective algebraic sets is a projective algebraic set.*
(3) *The empty set and $\mathbb{P}^n$ are projective algebraic sets.*

Exercise 2.13. Prove the Lemma. (Hint: It is very similar to what we did in affine space.)

We also need a way of getting a homogeneous ideal from a subset of $\mathbb{P}^n$:

Definition 2.14. Let $Y \in \mathbb{P}^n$. Define $I(Y) \subset k[x_0, \ldots, x_n]$ to be the ideal generated by the set

$$\{f \in k[x_0, \ldots, x_n] | f \text{ is homogeneous, and } f(p) = 0 \text{ for all } p \in Y.\}$$

Definition 2.15. We define the *Zariski topology* on $\mathbb{P}^n$ by taking the algebraic sets to be our closed sets.

This lets us define:

Definition 2.16. A *projective variety* is an irreducible closed subset of $\mathbb{P}^n$. A *quasi-projective variety* is an open subset of a projective variety. The dimension of a projective or quasi-projective variety is its dimension as a topological space.

Example 2.17. In $\mathbb{P}^2$, a *projective line* is given by $L = Z(f(x,y,z))$, where f is a homogeneous degree 1-polynomial. Let us show that any two distinct projective lines in $\mathbb{P}^2$ intersect each other precisely once.

Let

$$f_1(x,y,z) = ax + by + cz, \quad f_2(x,y,z) = dx + ey + fz$$

be two homogeneous degree 1-polynomials, and let $L_1 = Z(f_1), L_2 = Z(f_2)$. To find a point where they intersect, we solve the system of linear equations

$$ax + by + cz \quad = 0 \tag{2.1}$$

$$dx + ey + fz \quad = 0. \tag{2.2}$$

We note that if (x,y,z) is a solution of the system, so is $(\lambda x, \lambda y, \lambda z)$ for any nonzero λ - this is as we should expect, because $(x : y : z) = (\lambda x : \lambda y : \lambda z)$ in $\mathbb{P}^2$.

We have a system of two linear equations in three variables, and we know from linear algebra that this means there *must* be (at least) a one-dimensional family of solutions, which means that there is at least one point $p \in L_1 \cap L_2$.

Suppose that there is another point $q \in L_1 \cap L_2$, $q \neq p$. This means that the system (2.1) has a two-dimensional family of solutions, but this means that there is some $\mu \neq 0$ such that $(a,b,c) = (\mu d, \mu e, \mu f)$. But then $L_1 = L_2$.

In conclusion, any two distinct lines in $\mathbb{P}^2$ intersect in precisely one point.

For a more geometric proof for the same, consider that the points on a line in $\mathbb{P}^2$ correspond to the lines in k^3 passing through $\mathbf{0}$ that lie inside a plane. But any two planes in k^3, both containing $\mathbf{0}$, must intersect along a line containing $\mathbf{0}$. This line corresponds to the desired point in $\mathbb{P}^2$.

2.4. **The usefulness of glue.** How do we relate $\mathbb{A}^n$ to $\mathbb{P}^n$? We'll now see that $\mathbb{P}^n$ is covered by $n+1$ copies of $\mathbb{A}^n$.

Let $D^+(x_i) \subset \mathbb{P}^n$, for any $i \in \{0, \dots, n\}$, be the subset

$$D^+(x_i) = \{(a_0 : a_1 : \cdots : a_n) \mid a_i \neq 0\}.$$

Then $D^+(x_i)$ is a quasiprojective variety (why?)

If $p = (a_0 : \cdots : a_n)$ is a point in $\mathbb{P}^n$, at least one a_i must be nonzero, so $\mathbb{P}^n$ is covered by the $D^+(x_i)$.

If $p = (a_0 : \cdots : a_n) \in D^+(x_i)$, we can write

$$p = (a_0/a_i : \cdots : a_i/a_i : \cdots : a_n) = (a_0/a_i : \cdots : a_{i-1}/a_i : 1 : a_{i+1}/a_i : \cdots : a_n),$$

and we see that the ratios a_j/a_i are well-defined. But then we can define a function

$$\phi_i \colon \mathbb{A}^n \to D^+(x_i)$$

by

$$\phi_i \colon (b_1, \cdots, b_n) \mapsto (b_1 : \cdots : b_i : 1 : b_{i+1} : \cdots : b_n).$$

Lemma 2.18. *Consider $D^+(x_i)$ with its induced topology as a subspace of $\mathbb{P}^n$. Then the map $\phi_i \colon \mathbb{A}^n \to D^+(x_i)$ is a homeomorphism.*

Proof. It's enough to show that ϕ_i identifies the closed subsets of $D^+(x_i)$ with those of $\mathbb{A}^n$. Let's make things easy for ourselves and assume $i = 0$, writing ϕ for ϕ_0 and U for $D^+(x_0)$.

First, let $A = k[x_1, \ldots, x_n]$, $S = k[y_0, \ldots, y_n]$. Let S^{hom} be the set of homogeneous polynomials in S. We can then define maps $h \colon A \to S^{hom}$ by

$$h(f(x_1, \ldots, x_n)) = y_0^{\deg f} f(y_1/y_0, \ldots, y_n/y_0),$$

and $a \colon S^{hom} \to A$ by

$$a(g(y_0, \ldots, y_n)) = g(1, y_1, \ldots, y_n).$$

Exercise 2.19. Show that h indeed maps any polynomial to a homogeneous polynomial of the same degree.

Now, suppose that $Y \subset U$ is a closed set. Then its closure $\overline{Y} \subset \mathbb{P}^n$ is an algebraic set, so there is some set $T \subset S^{hom}$ of homogenenous polynomials such that $\overline{Y} = Z(T)$. We can check (do it!) that $\phi^{-1}(Y) = Z(a(T))$.

In the other direction, suppose that $X \subset \mathbb{A}^n$ is a closed set. Then $X = Z(T')$ for some subset $T' \subset A$. But we can again check (do it!) that $\phi(X) = Z(h(T')) \cap U$. This concludes the proof. □

Corollary 2.20. *Every projective variety $X \subset \mathbb{P}^n$ can be covered by quasiprojective varieties $X \cap D^+(x_0), \ldots, X \cap D^+(x_n)$ that are homeomorphic to affine varieties.*

We often call the $D^+(x_i)$ *affine charts* of $\mathbb{P}^n$, and the $D^+(x_i) \cap X$ *affine charts* of X.

2.5. **Computing projective closures and affine charts.** Suppose that $X \subset \mathbb{A}^n$ is an affine variety, and think of $\mathbb{A}^n$ as the open subset $D^+(x_n)$ of $\mathbb{P}^n$. Then we can try to compute $\overline{X} \subset \mathbb{P}^n$, and we call $\overline{X}$ the *projective closure* of X. The lemma 2.18 gives us a way to find $\overline{X}$. Let's show how to do it with some examples:

Example 2.21. Let $L_1 = Z(x_1)$, $L_2 = Z(x_1 - 1)$ be two lines in $\mathbb{A}^2$. Let $X = L_1 \cup L_2$, we find that $X = Z(x_1(x_1 - 1)) = Z(x_1^2 - x_1)$.

Now think of $\mathbb{A}^2$ as the subset $D^+(x_0)$ of $\mathbb{P}^2$, and we compute $\overline{X}$ by applying the map h to $(x_1^2 - x_1)$: We find $h(x_1^2 - x_1) = y_0^2((y_1/y_0)^2 - y_1/y_0) = y_1^2 - y_1y_0 = y_1(y_1 - y_0)$, so $\overline{X} = Z(y_1) \cup Z(y_1 - y_0)$.

Let's draw what $\overline{X}$ looks like on the three affine charts $D^+(x_0), D^+(x_1), D^+(x_2)$. We already know that $\overline{X} \cap D^+(x_0) = Z(x_1(x_1 - 1))$, but to find $\overline{X} \cap D^+(x_1)$ we apply the map a from lemma 2.18 (but with $i = 1$ instead of 0) to $y_1^2 - y_1y_0$. We find $a(y_1^2 - y_1y_0) = 1 - x_0$, so $\overline{X} \cap D^+(x_1) = Z(1 - x_0)$.

Finally, arguing similarly, we find that in the third chart $D^+(x_2)$, $\overline{X} \cap D^+(x_2) = Z(x_1(x_1 - x_0))$. Let's draw these (the figure is example 2.21, it's not here where I'd like it).

Notice that on the chart $D^+(x_1)$, we only see one line. Is it $\overline{L_1} \cap D^+(x_1)$ or $\overline{L_2} \cap D^+(x_1)$?

Example 2.22. Consider the degree 2-curve $C = Z(y_0y_1 - y_2^2) \subset \mathbb{P}^2$. (A degree 2-curve in the plane is often called a *conic section*, or just *conic.*) We compute its restrictions to each chart $D^+(x_i)$ similarly:

First, $D^+(x_0) \cap C = Z(x_1 - x_2^2)$, a parabola. Second, $D^+(x_1) \cap C = Z(x_0 - x_2^2)$, also a parabola. But $D^+(x_2) \cap C = Z(x_0x_1 - 1)$, a hyperbola. Exercise: draw these!

Remark 2.23. Suppose that $X \subset \mathbb{A}^n$ is an affine variety, and that $I(X)$ is generated by $(f_1, \ldots, f_r)$. It is *not* always true that the homogeneous ideal of $\overline{X}$ is generated by $h(f_1), \ldots, h(f_r)$.

It is, however, true if $r = 1$.

3. Curves

3.1. **Tangent lines.** Suppose that $C = Z(f) \subset \mathbb{P}^2$ is a curve, for some homogeneous $f \in k[x, y, z]$, and let $p = (a : b : c) \in C$. Let T_pC be the *tangent line* to C at p. How can we compute the equation of C?

Well, the line T_pC is by definition the best linear approximation to C at p, so we should get the linear terms of f. This motivates us to set

$$\ell = x\frac{\partial f}{\partial x}(p) + y\frac{\partial f}{\partial y}(p) + z\frac{\partial f}{\partial z}(p).$$

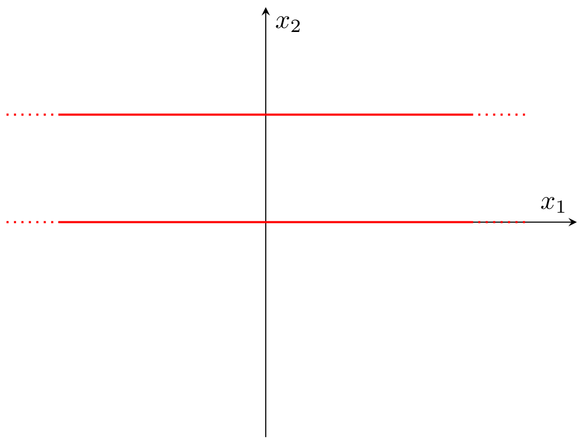
Then we have:

Idea 3.1. The tangent line to C at p is given by $T_pC = Z(\ell) \subset \mathbb{P}^2$.

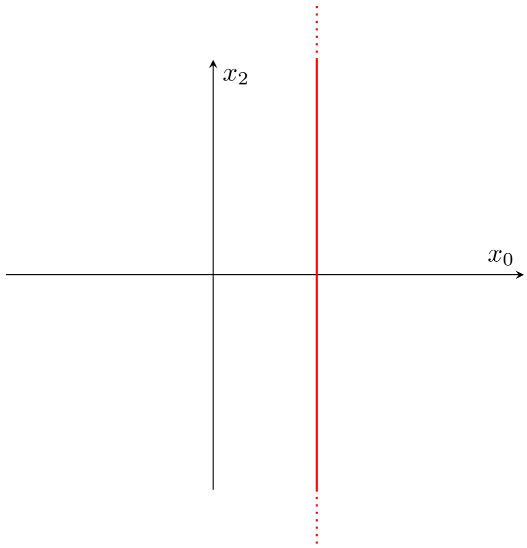
We can also do this affinely: Suppose $C = Z(f) \subset \mathbb{A}^2$. Then, at a point $p = (a, b) \in C$, the tangent line has equation

$$(x - a)\frac{\partial f}{\partial x}(p) + (y - b)\frac{\partial f}{\partial y}(p) = 0$$

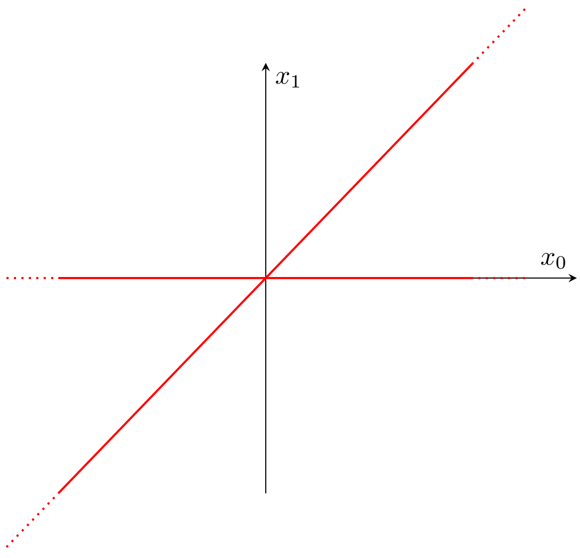
We can deduce this from the projective formula idea 3.1, using the following result of Euler:



(A) $\overline{X} \cap D^+(x_0)$ (coordinates x_1, x_2)



(B) $\overline{X} \cap D^+(x_1)$ (coordinates x_0, x_2)



(C) $\overline{X} \cap D^+(x_2)$ (coordinates x_0, x_1)

Lemma 3.2. *Suppose $f \in k[x_0, \dots, x_n]$ is homogeneous of degree d. Then $\sum_i x_i \frac{\partial f}{\partial x_i} = d \cdot f$.*

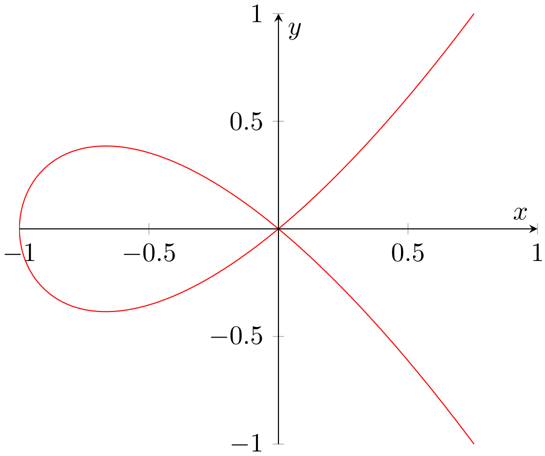
Suppose that $p = (a_0 : a_1 : a_2) \in D^+(a_2) \cap C \subset \mathbb{P}^2$, for some $f \in k[x_0, x_1, x_2]$ homogeneous of degree d. Then we can write $p = (a_0/a_2 : a_1/a_2 : 1)$, and setting $a = a_0/a_2, b = a_1/a_2$, we can rewrite idea 3.1 as

$$x\frac{\partial f}{\partial x}(p) + y\frac{\partial f}{\partial y}(p) + d \cdot f(p) - a\frac{\partial f}{\partial x}(p) - b\frac{\partial f}{\partial y}(p) = 0$$

which is what we want, since $f(p) = 0$. (I think the projective formula is both more beautiful and easier to remember!)

But it is not always possible to define a tangent line.

Consider the curve $C = Z(x^3 + x^2z - y^2z) \subset \mathbb{P}^2$. In the chart $D^+(z)$, it looks like



.

What should the tangent line at the origin $\mathbf{0}$ be? There is no obvious choice[5] If we try to compute the equation for the tangent line, we find that

$$\frac{\partial f}{\partial x}(\mathbf{0}) = \frac{\partial f}{\partial y}(\mathbf{0}) = \frac{\partial f}{\partial z}(\mathbf{0}) = 0,$$

so *there is no equation.* This motivates us to define

3.2. **Singularities.**

Definition 3.3. Let $C = Z(f) \subset \mathbb{P}^2$. If every partial derivative of f vanishes at P, we say that p is a *singular point* of C.

If $p \in C$ is not a singular point, we say that p is a *smooth point* of C. A curve is *smooth* if every point on it is smooth.

In fact, this is a special case of the more general

[5]Or, well, there are *two* choices. There are ways of formalising this intuition.

Definition 3.4. Let $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ be homogenenous polynomials. Let $X = Z(f_1, \dots, f_r) \subset \mathbb{P}^n$, and let $p \in X$. We say that X is *smooth at* p if the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_0}(p) & \dots, & \frac{\partial f_1}{x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_0}(p) & \dots, & \frac{\partial f_r}{x_n}(p) \end{bmatrix}$$

has rank $n - r$.

If the rank of the matrix is smaller than $n - r$, we say that X is *singular* at p.

We say that X is smooth if X is smooth at every point.

Now, this definition of smooth and singular points depends only on the geometry of X around p, so we can instead ask for a characterisation of singular points on an affine variety. Here it is:

Definition 3.5. Let $X = Z(f_1, \dots, f_k) \subset \mathbb{A}^n$ be an affine algebraic set of dimension r, for some polynomials $f_1, \dots f_k \in k[x_1, \dots, x_n]$. Let $p \in X$. Then we say that p is a *smooth* point of X if the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \dots, & \frac{\partial f_1}{x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}(p) & \dots, & \frac{\partial f_r}{x_n}(p) \end{bmatrix}$$

has rank $n-r$, and that p is a *singular* point of X if the matrix has rank smaller than $n-r$.

Again, X is smooth if X is smooth at every point.

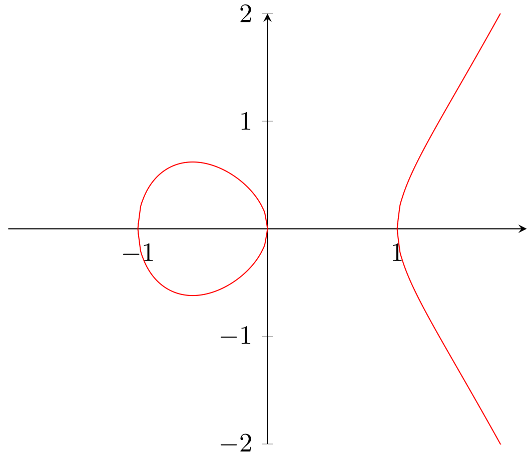
The matrix appearing in this definition is called the *Jacobian* matrix of X.

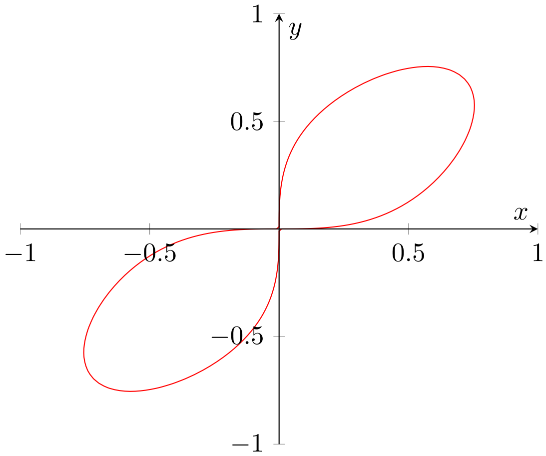
Remark 3.6. It may seem that these definitions of smoothness depend on the choice of coordinates of the affine and projective spaces – could it be that after performing a basis change, a smooth variety suddenly gains a singularity? This is, in fact, not possible, but it requires a fair bit of commutative algebra to prove properly.

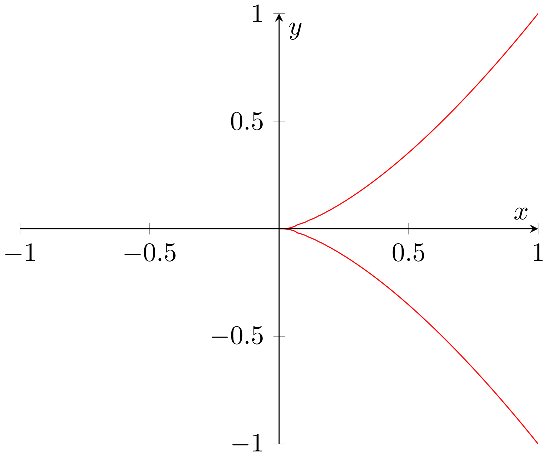
So this makes a fair bit of sense, and we should compute a few examples.

But one important note: Funny things can happen in characteristic p. For instance, if we let $f = x^p$, then $\frac{\partial f}{\partial x} = px^{p-1} = 0$ (since $p = 0$)!

3.3. **Smooth and singular curves.** Let's work in $\mathbb{A}^2$ with coordinates x, y, and draw some curves:







These pictures are of, respectively, the curves $Z(x^4 + y^4 - xy)$, $Z(y^2 - x^3 - x)$, and $Z(x^3 - y^2)$ in the plane.

Let's compute their singularities.

Exercise 3.7. We just do it. Pay special attention to computations when the characteristic of k is 2 or 3.

Example 3.8. It is possible for an equation to define a singular variety over some fields, but a smooth variety over others.

As an example, consider $C = Z(x^3 - y^2 - y) \subset \mathbb{A}^2_k$. Then C is smooth if $k = \mathbb{R}$ or $\mathbb{C}$ (for instance), but singular if k is any field with characteristic 3.

Let's also compute the singularities of a surface: Let $f = xy^2 - z^2 \in k[x, y, z]$, and set $S = Z(f)) \subset \mathbb{A}^3$. Then, S is singular along the locus where

- $xy^2 - z^2 = 0$,
- $y^2 = 0$,
- $2xy = 0$,
- $2z = 0$.

Solving all of these, we find that S is singular along the line $y = z = 0$, i.e., the x-axis.

4. Lecture 5: Morphisms of varieties and curves

(From now on, I'm going to assume that $k = \mathbb{C}$.)

We'd like to be able to say, for instance, that any two lines are "the same", or *isomorphic*. But we still don't have a way of actually doing that. So we'll introduce the idea of a *morphism*.

Suppose that we have two algebraic sets, $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$. We want to describe a function $\phi\colon X \to Y$, that should work well with the rest of the theory. Everything we've done so far has been with polynomials, so let's try that:

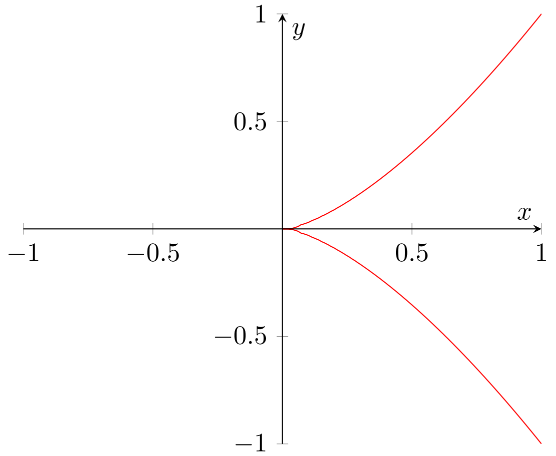
Definition 4.1. A function $\phi\colon X \to Y$ is a *morphism* if there are polynomials $f_1, \ldots f_m \in k[x_1, \ldots, x_n]$ such that

$$\phi(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$$

for all points $(a_1, \ldots, a_n) = p \in X$.

Example 4.2. Let $X = Z(x^2 + y^2 - 1) \subset \mathbb{A}^2$, and define $\phi\colon X \to \mathbb{A}^1$ by $\phi(a, b) = a$. Then this is a morphism, and we can understand it as projection onto the x-axis.

Example 4.3. Consider a morphism $\phi\colon \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ given by $\phi(t) = (t^2, t^3)$. Then this is a morphism, and the image is the curve $C = V(y^2 - x^3)$. (This curve is usually called the *cuspidal cubic*). It looks like



Example 4.4. Let's do a very useful example: An *affine transformation* of $\mathbb{A}^n$ is any function $\phi\colon \mathbb{A}^n \to \mathbb{A}^n$ for which there exist two $n \times n$-matrices A, B such that A is invertible, and such that ϕ can be written as

$$\phi\colon (a_1, \dots, a_n) \mapsto A \cdot (a_1, \dots, a_n)^T + B.$$

(Here ^{T} signifes the transpose.)

Then ϕ is a morphism.

Exercise 4.5. Let $p \in \mathbb{A}^n$, $p \neq \mathbf{0}$. Find an affine transformation $\phi\colon \mathbb{A}^n \to \mathbb{A}^n$ such that $\phi(p) = \mathbf{0}$.

We now prove:

Lemma 4.6. *A morphism is continuous for the Zariski topology.*

Proof. This comes down to showing that the inverse image of a closed set under a morphism is closed. So let $\phi\colon X \to Y$ be a morphism, and let $Y' \subset Y \subset \mathbb{A}^m$ be closed. Then $Y' = Z(S)$ for some set of polynomials $S \subset k[x_1, \dots, x_m]$. Write

$$\phi\colon (a_1, \dots, a_n) = (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n))$$

for some polynomials $f_i \in k[x_1, \dots, x_n]$.

Suppose then that $p = (a_1, \dots, a_n) \in \phi^{-1}(Z)$: this means that, for every $g \in S$, we have

$$0 = g(\phi(p)) = g(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)).$$

Now, for any $g(x_1, \dots, x_m) \in S$, we set

$$\phi^* g = g(f_1(x_1, \dots, x_n), \dots f_n(x_1, \dots, x_n)).$$

then, for every $g \in S$, we have

$$\phi^* g(p) = g(f_1(x_1, \dots, x_n), \dots f_n(x_1, \dots, x_n))|_{x_1=a_1,\dots,x_n=a_n} = 0.$$

If we set $\phi^* S = \{\phi^* g \mid g \in S\}$, this shows that $Z(\phi^* S) \supset \phi^{-1}(Y')$. The other inclusion can be shown by doing this argument in reverse.

Therefore $\phi^{-1}(Y') = Z(\phi^* S)$, so $\phi^{-1}(Y')$ is an algebraic set.

□

Let's introduce some notation: For any algebraic sets $X \subset \mathbb{A}^n,\ Y \subset \mathbb{A}^m$, set

$$\mathrm{Hom}(X, Y)$$

to be the set of all morphisms $\phi\colon X \to Y$.

Recall that we defined, for any algebraic set $X \subset \mathbb{A}^n$, the ring $\mathcal{O}(X) = k[x_1, \dots, x_n]/I(X)$.

Now let $\phi\colon X \to Y$ be a morphism. The proof of the lemma above shows that $\phi^*(I(Y)) \subset I(X)$. But this means that ϕ induces a *ring homomorphism* $f^\sharp\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ as follows: Let $f \in \mathcal{O}(Y)$, then pick some $\overline{f} \in \mathcal{O}(\mathbb{A}^m) = k[x_1, \dots, x_m]$ such that $f = \overline{f} + I(Y)$. Then

$$\phi^*(\overline{f} + I(Y)) = \phi^*(\overline{f}) + \phi^*(I(Y)) \subset \phi^*(\overline{f}) + I(X).$$

We *define*

$$\phi^\sharp(f) = \phi^*(\overline{f}) + I(X) \in \mathcal{O}(X).$$

Exercise 4.7. Show that $\phi^\sharp$ is a well-defined ring homomorphism. (First show that $\phi^\sharp$ is a well-defined function. *Well-defined* means the following: the element $\phi^\sharp(f)$ is independent of the choice of $\overline{f}$. Then show that $\phi^\sharp$ is a ring homomorphism.)

We now wish to show:

Theorem 4.8. *Let $X \subset \mathbb{A}^n,\ Y \subset \mathbb{A}^m$ be algebraic sets. There is a bijection*

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$$

given by $\phi \mapsto \phi^\sharp$. Here $\mathrm{Hom}(X, Y)$ is the set of all morphisms from X to Y, and $\mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$ is the set of all ring homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.

Proof. We've just seen how to get a ring homomorphism $\phi^\sharp$ from a morphism ϕ of algebraic sets.

Now suppose that $\psi\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a ring homomorphism. We're going to build a morphism $X \to Y$. Start by writing $\mathcal{O}(Y) = k[y_1, \dots y_m]/I(Y)$. Let, for every i, y_i' be the image of y_i in $\mathcal{O}(Y)$. Set $\xi_i = \psi(y_i') \in \mathcal{O}(X)$. Then we can think of the ξ_i as functions $X \to \mathbb{A}^1 = k$, so we can define a map $\sigma\colon X \to \mathbb{A}^m$ by

$$\sigma(P) = (\xi_1(P), \dots, \xi_m(P)).$$

We're now going to show that Y contains the image of σ. Since $Y = Z(I(Y))$, it's enough to show that $f(\sigma(P)) = 0$ for any $f \in I(Y)$ and for any $P \in X$.

So we compute

$$f(\sigma(P)) = f(\xi_1(P), \dots, \xi_m(P)) = f(\psi(y'_1)(P), \dots, \psi(y'_m)(P)) = \psi(f(y_1), \dots, f(y_m))(P) = 0.$$

That σ is a morphism is clear from the construction. □

A morphism $\phi\colon X \to Y$ such that there is another morphism $\psi\colon Y \to X$ with $\psi \circ \phi = \mathrm{id}_X,\ \phi \circ \psi = \mathrm{id}_Y$, is called an *isomorphism*.

From the theorem just proved, we immediately have:

Corollary 4.9. *Two algebraic sets X, Y are isomorphic if and only if the rings $\mathcal{O}(X), \mathcal{O}(Y)$ are isomorphic.*

Example 4.10. Any two lines are isomorphic. To see this, simply note that if $L \subset \mathbb{A}^n$ is a line, then

$$\mathcal{O}(L) = k[x_1, \dots, x_n]/(l_1, \dots, l_{n-1}) = k[x]$$

, where the l_i are linearly independent linear functions in the $\{x_i\}$.

Example 4.11. The parabola $Z(x^2 - y) \subset \mathbb{A}^2$ is isomorphic to a line: we have $k[x, y]/(x^2 - y) \cong k[x]$.

Example 4.12. Let $C \subset \mathbb{A}^2$ be given by $C = Z(x^2 - y^3)$, the cuspidal cubic. The morphism $\phi\colon C \to L = Z(y)$ given by $C \ni (a, b) \mapsto a \in L$ is *not* an isomorphism: the morphism ϕ corresponds to the ring homomorphism $\phi^\sharp\colon k[x] \to k[x, y]/(x^2 - y^3)$ given by $x \mapsto x$. But $\phi^\sharp$ is not an isomorphism, because there is no $f \in k[x]$ such that $\phi^\sharp(f) = y$.

Indeed, there is *no* isomorphism $k[x] \cong k[x, y]/(x^2 - y^3)$. To see this, it is enough to note that $k[x]$ is a principal ideal domain, and the ideal $(x, y) \subset k[x, y]/(x^2 - y^3)$ is not principal. (A principal ideal domain is an integral domain in which every ideal is principal, that is, generated by one element.)

Exercise 4.13. Consider the morphism $a\colon L \to C$ given in example 4.3. What is the corresponding ring homomorphism

$$a^\sharp\colon k[y_1, y_2]/(y_1^2 - y_2^3) \to k[x]?$$

4.1. **Regular functions.** In general, we want a notion of morphism that includes projective, quasiaffine, and quasiprojective varieties. To get that, we have to work a little bit. Let $U \subset X$ be an open subset of an affine variety $X \subset \mathbb{A}^n$. Consider a function $f\colon U \to k$: we say that f is *regular* if there are polynomials $g, h \in k[x_1, \dots, x_n]$, such that h is nowhere zero on U, and such that $f = \frac{g}{h}$ everywhere on U.

Similarly: Let $V \subset Y$ be an open subset of a projective variety $Y \subset \mathbb{P}^n$. Let $f\colon V \to k$ be a function: we say that f is *regular* if there are *homogeneous polynomials of the same degree* $g, h \in k[x_0, \ldots, x_n]$ such that h is nowhere zero on V, and $f = \frac{g}{h}$ everywhere on V.

Definition 4.14. Let *variety* mean any of quasi-affine, quasi-projective, affine, and projective variety.

A *morphism* of varieties is a continuous function $\phi\colon X \to Y$ such that, for every open subset $V \subset Y$ and every regular function $f\colon V \to k$ on V, the composition $f \circ \phi$ is a regular function on $\phi^{-1}(V)$.

We can check that this definition of *morphism* includes the morphisms of affine varieties defined above (exercise!)