ALGEBRAIC GEOMETRY EXERCISES

1. Affine varieties

(1) Draw the following algebraic sets in $\mathbb{A}^2_{\mathbb{R}}$ (with coordinates x, y):
   (a) $Z(xy-1)$
   (b) $Z(x^3 - xy^2)$
   (c) $Z(x^2 + y^2 - 1)$

(2) Show that the set of points $\{(t, \sin t) \mid t \in \mathbb{R}\} \subset \mathbb{A}^2_{\mathbb{R}}$ is not an algebraic set.

(3) Let $X = Z(I) \subset \mathbb{A}^n$ for some ideal $I \subset k[x_1, \ldots, x_n]$. Let $p \in \mathbb{A}^n \setminus X$. Show that there is a polynomial $f \in I$ such that $f(p) = 1$.

(4) Show that $f = x^2 + y^2 + z^2$ is irreducible as an element of $\mathbb{R}[x, y, z]$. What is the dimension of $Z(f) \subset \mathbb{A}^3_{\mathbb{R}}$?

(5) Let $X \subset \mathbb{A}^3$ be the algebraic set defined by $X = Z(x^3 + y^2 + z^2, xy - xz)$. Describe the irreducible components of X.

(6) Show that the algebraic set $Z(xy, xz) \subset \mathbb{A}^3$ has two components of different dimension.

(7) (a) Work in the affine plane $\mathbb{A}^2_{\mathbb{R}}$, with coordinates x, y. Let $L = Z(y)$ be the x-axis, choose a number t and let $Q_t = Z(x^2 + t - y)$ be the parabola. Determine the subsets U_0, U_1, U_2 of $\mathbb{R}$ such that:

$$L \cap Q_t = \begin{cases} \emptyset \text{ if and only if } t \in U_0 \\ \text{a point if and only if } t \in U_1 \\ \text{two points if and only if } t \in U_2 \end{cases}$$

   (b) Which of U_0, U_1, U_2 are quasi-affine varieties?
   (c) Do (a) and (b) again, but with $\mathbb{C}$ in place of $\mathbb{R}$.

(8) (a) Suppose that L is a line in $\mathbb{A}^2$, and let $C \subset \mathbb{A}^2$ be a curve of degree $d > 1$. Show that $L \cap C$ consists of at most d points.
   (b) Construct an example to show that it is possible for $L \cap C$ to be empty, even when working over $k = \mathbb{C}$.

(9) (a) Let $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ be a point. Show that $I(\{p\}) = (x_1 - a_1, \ldots, x_2 - a_2, \ldots, x_n - a_n)$.
   (b) Show that $I(\{p\})$ is a maximal ideal of $k[x_1, \ldots, x_n]$.
   (c) Show that, if $k = \mathbb{C}$, every maximal ideal of $k[x_1, \ldots, x_n]$ has the form $I(\{p\})$.

(d) Give an example of a maximal ideal of $\mathbb{R}[x]$ that is *not* of the form $I(\{p\})$.

(10) For every polynomial $f \in k[x_1, \dots, x_n]$, let $D(f) = \mathbb{A}^n \setminus Z(f)$. Show that the sets $D(f)$ form a *subbasis* for the Zariski topology on $\mathbb{A}^n$. (This means: Any open set is a union of finite intersections of sets on the form $D(f)$.)

(11) *The twisted cubic curve.* Consider the set of points $C = \{(t, t^2, t^3) | t \in k\} \subset \mathbb{A}^3$. Find generators of the ideal $I(C)$, and show that C is an affine variety. Show that there is an isomorphism $k[x] \cong \mathcal{O}(C)$.

(12) Think of $\mathbb{A}^2$ as $\mathbb{A}^1 \times \mathbb{A}^1$ in the straightforward way. Show that the Zariski topology on $\mathbb{A}^2$ does not equal the product of the Zariski topologies on the two copies of $\mathbb{A}^1$.

(13) Show that a nonempty open subset U of an irreducible topological space X is irreducible and *dense*. (U is dense if every open subset $V \subset X$ satisfies $U \cap V \neq \emptyset$.)

(14) Let B be a finitely-generated k-algebra. Show that there is some algebraic set V for which $B = \mathcal{O}(V)$ if and only if B has no nilpotent elements. (A *nilpotent* element of a ring B is an element $a \in B$ such that $a \neq 0$, and there is some $k > 0$ such that $a^k = 0$.)

2. Projective varieties

(1) Find the projective coordinates of every point in
  (a) $X = Z(x_0 - 3x_1, x_0^2 + x_1^2) \subset \mathbb{P}^1_{\mathbb{R}}$
  (b) $X = Z(2x_0^4 - x_1^4 - x_2^4, x_1 - x_2) \subset \mathbb{P}^2_R$
  (c) $X = Z(2x_0^4 - x_1^4 - x_2^4, x_1 - x_2) \subset \mathbb{P}^2_{\mathbb{C}}$
  (d) $X = Z(x_0x_1 + x_1x_2 + x_0x_2, x_0x_1x_2) \subset \mathbb{P}^2_{\mathbb{R}}$

(2) Let $X = Z(x_0) \subset \mathbb{P}^2$. Show that the affine charts $X \cap D^+(x_1), X \cap D^+(x_2)$ are both lines in the affine plane. Show that $X \cap D^+(x_0) = \emptyset$. (For this reason, we sometimes call X the "line at infinity").

(3) Let $X = Z(x_0x_1 + x_2^2) \subset \mathbb{P}^2$. Show that $X \cap D^+(x_0)$ and $X \cap D^+(x_1)$ are both parabolas. Show that $X \cap D^+(x_2)$ is a hyperbola.

(4) Consider the projective algebraic set $C = Z(x^3 + y^3 + xyz) \subset \mathbb{P}^2$. Before drawing them, in which two of the affine charts $C \cap D^+(x), C \cap D^+(y), C \cap D^+(z)$ will you get the same image? Draw each of the charts to verify. (Use a computer if you like!)

(5) (a) Show that $X = Z(xy - z^2) \subset \mathbb{P}^2$ is the projective closure of the hyperbola $X' = Z(xy - 1) \subset \mathbb{A}^2$.
  (b) Let $L_1', L_2' \subset \mathbb{A}^2$ be the asymptotic lines to X'. Describe the projective closures L_1, L_2 of the two lines. Show that $X \cap L_1$ and $X \cap L_2$ are nonempty.

(6) Count the number of points in $\mathbb{P}^n_{\mathbb{F}_q}$, where $\mathbb{F}_q$ is the finite field with q elements. Conclude that the only n, m for which $\mathbb{P}^n_{\mathbb{F}_q} = \mathbb{A}^m_{\mathbb{F}_q}$ are $n = m = 0$.

(7) Show that every quasiprojective variety can be covered by quasiaffine varieties.

(8) If $f \in k[x_0, \dots, x_n]$ is homogeneous of degree 1, we call $H = Z(f) \subset \mathbb{P}^n$ a *hyperplane*. Show that the intersection of n hyperplanes in $\mathbb{P}^n$ is nonempty. Show that if the intersection is a finite set, then it is precisely one point.

(9) Show that if $H \subset \mathbb{P}^n$ is a hyperplane, there is a homeomorphism $H \cong \mathbb{P}^{n-1}$.

(10) (a) Show that if $k = \mathbb{C}$, the intersection of a curve and a line in $\mathbb{P}^2$ is nonempty.
(b) If $f \in k[x_0, \dots, x_n]$ is irreducible and homogeneous of degree d, we call $X = Z(f) \subset \mathbb{P}^n$ a *projective hypersurface* of degree d. Show that if $k = \mathbb{C}$, the intersection of X with $n-1$ hyperplanes is nonempty. (Hint: Use induction, with (a) as the base case.)

(11) Consider the function $\phi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$\phi : ((a_1 : a_2), (b_1 : b_2)) \mapsto (a_1 b_1 : a_1 b_2 : a_2 b_1 : a_2 b_2).$$

(a) Show that ϕ is injective, and that the image of ϕ is the projective variety $X = Z(x_0 x_3 - x_1 x_2)$.
(b) Show that for any $p \in \mathbb{P}^1$, both $\phi(p \times \mathbb{P}^1)$ and $\phi(\mathbb{P}^1 \times p)$ are lines. Show that every line in X can be written as either $\phi(p \times \mathbb{P}^1)$ or $\phi(\mathbb{P}^1 \times p)$.
(c) Show that, if l_1, l_2 are two lines in X, then it is possible that $l_1 \cap l_2 = \emptyset$. When does this happen?

(12) Let $X = Z(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3_{\mathbb{C}}$. Show that there are at least 27 lines in X. (In fact, there are precisely 27, but we won't show this.)

(13) Choose an integer $n > 0$ and consider the function $\phi\colon \mathbb{P}^1 \to \mathbb{P}^n$ given by

$$(a_0 : a_1) \mapsto (a_0^n : a_0^{n-1} a_1 : \cdots : a_0 a_1^{n-1} : a_n^n).$$

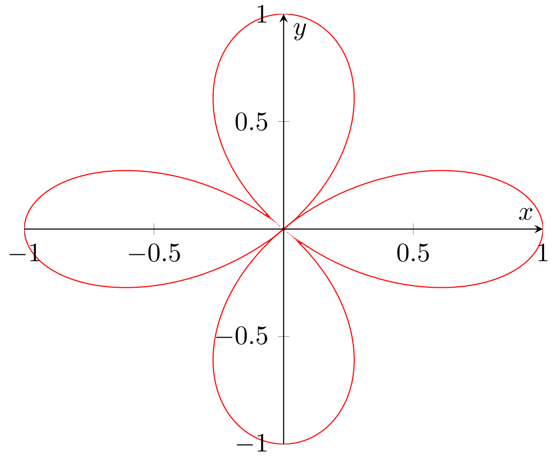
(a) Show that ϕ is injective.
(b) Show that the image of ϕ is a projective variety, and find generators for $I(\mathrm{im}(\phi))$.
(c) Show that, for $n = 3$, $\phi(\mathbb{P}^1)$ is the projective closure of the *twisted cubic curve* from exercise (11) in the *Affine varieties* section.

3. Curves and singularities

(1) Draw the following curves in the affine plane $\mathbb{A}^2_{\mathbb{R}}$. (Use a computer if you want!) If they are singular, find their singular points.
(a) $Z(x^3 - y^2 - 1)$
(b) $Z(x^2 - y^2 - y^3)$

(c) $Z(x^3 - y^3 - yx)$. (This curve is called the *folium of Descartes.*) Find the equation of the unique line that does not intersect the curve.

(d) $Z((x^2+1)y-1)$. (This curve is called the *witch of Agnesi.*)



(2) Show that the degree of this curve:

is at least 6. (Hint: Find a line that intersects the curve 6 times.) (The equation is $(x^2+y^2)^3 - (x^2-y^2)^2$.)

(3) Draw the curve $C = Z((x^2+y^2-1)^3 - x^2y^3) \subset \mathbb{A}^2_{\mathbb{R}}$. (Isn't it nice?) Where are the singular points of C?

(4) Let X be a (projective or affine) variety. Let $\mathrm{Sing}(X)$ be the set of singular points on X. Show that $\mathrm{Sing}(X)$ is a closed proper subset of X. We call $\mathrm{Sing}(X)$ the *singular locus* of X.

(5) Find the singular locus of the projective variety

$$Z(x_0x_1x_2 + x_0x_2x_3 + x_0x_1x_3 + x_1x_2x_3) \subset \mathbb{P}^3.$$

(This variety is called the *Cayley nodal cubic surface.*)

(6) Choose an integer n.

(a) Show that the surface $S = Z(x_1x_2 - x_3^n) \subset \mathbb{A}^3$ is singular for every $n > 1$.

(b) Identify $\mathbb{A}^3$ with $D^+(x_0) \subset \mathbb{P}^3$, and find the equations for the projective closure $\overline{S}$ of S in $\mathbb{P}^3$. Where is $\overline{S}$ singular?

(7) Let $C = Z(x^2+yz) \subset \mathbb{P}^2_k$, where k is a field of characteristic 2. Show that, for any point $p \in C$, the tangent line T_pC passes through the point $(1:0:0)$. (We say that C is a *strange curve.*)

4. Morphisms

(1) *Projection from a point.* Let $p = (1,0,0,\ldots,0) \in \mathbb{A}^n$, and let $H = Z(x_1)$ be the hyperplane. Define a function $\pi\colon \mathbb{A}^n \setminus \{p\} \to H$ by sending any point $q = (a_1, a_2, \ldots, a_n)$ to $(0, a_2, \ldots, a_n) \in H$. Show that π is a morphism of varieties.

(2) Let X be the cylinder $Z(x^2+y^2-1) \subset \mathbb{A}^3$, and let Y be the double cone $Z(x^2+y^2-z^2) \subset \mathbb{A}^3$.

(a) Show that the function $\mathbb{A}^3 \to \mathbb{A}^3$ given by

$$(a,b,c) \mapsto (ac, bc, c)$$

restricts to a function $\phi\colon X \to Y$. Show that ϕ is a morphism of affine varieties.

(b) Describe the morphism ϕ geometrically. Why is there no morphism $\psi\colon Y \to X$ such that $\psi \circ \phi$ is the identity?

(3) Let p be a prime number, and let k be a field of characteristic p.

(a) Show that there is a ring homomorphism $\phi\colon k[x_1,\dots,x_n] \to k[x_1,\dots,x_n]$ determined by $f \mapsto f^p$ for every $f \in k[x_1,\dots,x_n]$.

(b) Let $X \subset \mathbb{A}^n_k$ be an affine variety. Show that ϕ induces a morphism $\phi'\colon X \to X$. (This is called the *Frobenius endomorphism.*)

(4) (a) Let $f \in k[x_1,\dots,x_n]$ be a nonzero polynomial. Let $U = \mathbb{A}^n \setminus Z(f)$. Show that U is isomorphic to the affine algebraic set

$$Y = Z(x_{n+1}f - 1) \subset \mathbb{A}^{n+1}.$$

(Hint: You can define a map $\phi\colon U \to Y$ by

$$(a_1,\dots,a_n) \mapsto (a_1,\dots,a_n, \frac{1}{f(a_1,\dots,a_n)}).$$

Show that ϕ is the required isomorphism.)

(b) Use (a) to show that, if $X \subset \mathbb{A}^n$ is an affine variety, then $X \setminus Z(f)$ is isomorphic to an affine variety.

(5) Let $f \in k[x_0,\dots,x_n]$ be irreducible and homogeneous of degree $d > 1$. Let $X = Z(f) \subset \mathbb{A}^{n+1}$, and let $Y = Z(f) \subset \mathbb{P}^n$. Show that $\mathbf{0} \in X$, and that there is a morphism $X \setminus \mathbf{0} \to Y$ given by $X \ni (a_0,\dots,a_n) \mapsto (a_0 : \cdots : a_n) \in Y$. We call X the *cone* over Y.

(6) Let C be a projective curve, let $U \subset C$ be open, and let $f\colon U \to \mathbb{A}^1$ be a regular map. Assume that there is no other subset $U' \subset C$ such that a) $U \subsetneq U'$ b) there is a regular map $f'\colon U' \to \mathbb{A}^1$ such that $f'|_U = f$. Show that there is another open subset $V \subset C$ and a regular map $g\colon V \to \mathbb{A}^1$ with the properties that:

- $V \cup U = C$
- On $V \cap U$, $f = \frac{1}{g}$.

Use f and g to define a morphism $C \to \mathbb{P}^1$. (Hint: Write $\mathbb{P}^1$ as a union of $D^+(x_0)$ and $D^+(x_1)$, and consider f and g as morphisms to $D^+(x_0), D^+(x_1)$, respectively.)

Conclude that for every curve C, it is always possible to find a morphism $C \to \mathbb{P}^1$.

(7) *The function field of a variety.* Let X be a variety, and consider the set $\mathrm{Reg}(X)$ of all pairs (V, f) where V is an open subset of X and $f\colon V \to k$ is a regular function. We impose an equivalence relation $\sim$ on $\mathrm{Reg}(X)$ by saying:

$$(V, f) \sim (U, g) \Leftrightarrow f|_{U\cap V} = g|_{U\cap V}.$$

Let $K(X)$ be the set of $\sim$-equivalence classes of $\mathrm{Reg}(X)$. For any $(V, f) \in \mathrm{Reg}(X)$, we write its $\sim$-equivalence class as $\overline{(V, f)}$. In this exercise, we will show that $K(X)$ is a field.

(a) Let $(V, f), (U, g)$ be two elements of $\mathrm{Reg}(X)$. Show that $V \cap U$ is nonempty. Then:

(b) Show that $\overline{(V \cap U, f + g)} \in K(X)$.

(c) Show that $\overline{(V \cap U, fg)} \in K(X)$.

(d) For any $\overline{(V, f)} \in K(X)$, show that $\overline{(V, -f)} \in K(X)$.

(e) Let $\overline{(V, f)} \in K(X)$, and suppose that we can write $f = g/h$ where g, h are polynomials. Show that $\overline{(V \setminus Z(g), \frac{1}{f})} \in K(X)$.

(f) Use all the previous points to define a field structure on $K(X)$.

(g) Show that, if $U \subset X$ is open, then there is an isomorphism of fields $K(U) \to K(X)$.

(h) Show that $K(\mathbb{A}^1) = k(x)$, the field of all rational functions in one variable. Conclude that $K(\mathbb{P}^1) = k(x)$.

The field $K(X)$ is called the *function field* of X.